SMT-Final Algebra: Functions

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A brief introduction to functional equations. Adapted from Mathematical Buffet (2016) by Victor Ufnarovski, Jana Madjarova, and Frank Wikström. The lecture contains problems that require full solutions and questions that mainly require proofs up to the reader's own satisfaction. Some suggestions and problems are taken from Evan Chen's 2016 compendium on functional equations. I have sometimes left space for answers, which may come in handy for the reader later in the lecture. For more reading, and many more examples, I refer to Wu Pang-Cheng's 2018 compendium on Art of Problem Solving

Definitions

Before we start working with functions we need to agree on what they are. Per definition, a function consists of a set X called the *domain* (sv. domän), a set Y called the *codomain* (sv. codomän), and some rule $f(x)$ for every $x \in X$ defining a unique element $y \in Y$. Often, when we talk about real-valued functions, we are lazy and hope that the domain is inferred from the rule f while assuming the codomain is $\mathbb{R}$. For example, if we discuss the function $\sqrt{x}$ we typically mean

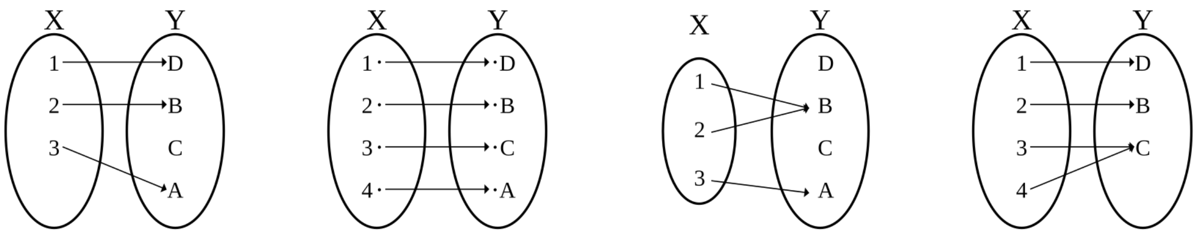
$$f : \mathbb{R}^+ \to \mathbb{R},\ x \mapsto \sqrt{x}$$

To help us understand the properties of functions, we typically divide functions into *surjective*, *injective*, and *bijective*.

Q.1: What is a surjective function?

Q.2: What is an injective function?

Q.3: What is a bijective function?



(a) (b) (c) (d)

Figure 1

Q.4: What are each of the functions defined in Figure 1?

Q.5: Draw the graph of two different real-valued, one-dimensional functions for each type in Figure 1. Ask your peer to specify which is which.

Q.6: Classify the following as bijective, surjective, injective or neither:

$$f_1 : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2 \qquad f_2 : \mathbb{R} \to \mathbb{R}^+,\ x \mapsto x^2$$
$$f_3 : \mathbb{R}^+ \to \mathbb{R},\ x \mapsto x^2 \quad f_4 : \mathbb{R}^+ \to \mathbb{R}^+,\ x \mapsto x^2$$

We see that the domains are as crucial to the properties of a function as is the actual rule. Only when a function is known to be bijective is the inverse $f^{-1} : Y \to X$ defined.

Most functions that we are used to working with have nice (and very strong) properties. They are typically *differentiable* (sv. deriverbara), *continuous* (sv. kontinuerliga) and sometimes even *montone* (sv. monotona).

- A function is **differentiable** at a point c if $f'(c)$ exists. If f is differentiable in all c in the domain, then f is differentiable.
- A function is **continuous** in a point c if $\lim_{x \to c} f(x) = f(c)$. If f is continuous in all c in the domain, then f is continuous.
- A function is **monotone** if it "perserves the order" of the domain. We say monotonically increasing if $x \leq y \implies f(x) \leq f(y)$ and conversely monotonically decreasing if $x \leq y \implies f(x) \geq f(y)$. When the inequality is strict, we talk about "stricly monotone".

A common problem for new students approaching functional equations and functions in general is that they underestimate the sheer number of possible functions. It is often tempting to assume a function is a polynomial, differentiable, monotone, or continuous. To combat this temptation it is useful to get accustomed to some seemingly strange functions.

Q.7: Draw, or attempt to draw, the following functions:

1. $f_1(x) = 1/x, x > 0$
2. $f_2 : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad x \mapsto \dfrac{x}{|x|}$
3. $f_3(x) = \lfloor x \rfloor$
4. $f_4(x) = x - \lfloor x \rfloor$
5. $\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$
6. $f_5 : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$
7. $f_6(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q}, \ \gcd(p, q) = 1, \ q > 0 \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

which of these are monotone, differentiable, and / or continuous?

Let's conclude this discussion about strange functions with a classical example of a functional equation that will trap you in a false sense of security.

Problem 1:
Find all monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that $(f(x))^2 = 1$

Finding optima

We commonly talk about *minimum* as the smallest point in a set such as the codomain of a function. But this is often not sufficient.

Q.8: Whatis the minimum of $f : \mathbb{R}^+ \to \mathbb{R} \ x \mapsto 1/x$?

For this reason, we often choose to instead talk about *supremum* and *infimum*:

$$\sup\{f(x)\} = \min_y \{y \geq f(x), \ \forall x\}$$

$$\inf\{f(x)\} = \max_y\{y \leq f(x),\ \forall x\}$$

Q.9: Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = x$. Find $\inf f$, $\sup f$, $\min f$, and $\max f$ if they exist.

Q.10: Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $\sup f = 1$ but f has no maximum. Compare with $g(x) = \sin x$, which satisfies $\sup g = \max g = 1$. What is the key difference?

Q.11: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Does f have an infimum? A minimum? What changes if f is instead defined on $(0, \infty)$?

Q.12: True or false: *"If f has a minimum, then $\min f = \inf f$."* Give an example of a function that has an infimum but no minimum, and one where $\inf f = -\infty$.

Theorem 1

Let $f : [a, b] \to \mathbb{R}$ be a *continuous* function. Then

- f is bounded and attains its maximum and minimum values.
- f attains all values between the minimum m and the maximum M.
- f attains all values between $f(a)$ and $f(b)$, a result called the **intermediate value theorem**.
- If $f(a)f(b) \leq 0$, there exists a $f(c) = 0$.
- if f can take all values in $[a, b]$, there exists a fixed point $x = f(x)$.

Q.13: Find an example of an open function that violates the first statement in Theorem 1.

Having defined the maximum and minimum, we need to understand how we find them. Finding the point where $f'(x) = 0$ is not enough.

- The function $f : X \to \mathbb{R}$ has a (global) minimum at the point x_0 if $f(x_0) \leq f(x),\ \forall x \in X$. We similarly define the maximum.
- The function f has a local minimum at a point x_0 if there exists an open interval $I = (a, b)$ containing x_0 such that $f(x_0)$ is a global minimum on I. We call all local minimum and maximum *extremum.*
- If the domain X is open and f is differentiable, $f'(x_0) = 0$ at every extremum. However, the point $f'(x^*) = 0$ does not need to be an extremum.
- If f is differentiable in x_0, $f'(x_0) = 0$, and $f''(x_0) > 0$, then $f(x_0)$ is a local minimum. We similarly find the local maxima.
- If X is a closed interval $[a, b]$, f is differentiable on (a, b) and continuous on $[a, b]$, then f may also have extrema at $x = a$ and $x = b$.
- The global minimum of a continuous, differential function f on a closed interval $X = [a, b]$ is found by evaluating the function at $f(a), f(b)$, and $f(x_i), f'(x_i) = 0$.

Q.14: Find the maximum point of $f : [1, 2] \to \mathbb{R}\ x \mapsto x^2$

Q.15: Find the maximum of $f : [-1, 1] \to \mathbb{R}\ x \mapsto x^2$

Q.16: Find the maximum of $f : (-1, 1) \to \mathbb{R}\ x \mapsto x^2$

Q.17: Convince a friend: "If f is differentiable in (a, b) and the equation $f'(x) = 0$ has *one* solution $x_0 \in (a, b)$; then f has a global minimum in x_0 if $f''(x_0) > 0$."

Q.18: Find the minimum of $f : (-1, 1) \to \mathbb{R}\ x \mapsto |x|$. What rules did you use?

Commonly used operations with functions

If we can add, subtract, multiply or divide in the codomain Y, we can also do so with functions $f : X \to Y$. For example

$$(f+g)(x) = f(x) + g(x)$$

You have likely encountered compositions of functions, $h(g(x))$, when working with derivatives. We can formalise this by stating that if $g : X \to Y$, and $h : Y \to Z$, then $f : X \to Z,\ f(x) = h(g(x))$. We commonly write this as

$$f = h \circ g$$

pronounced "composed with" or "circle" (sv. "sammansatt med" eller "boll"). Note that the operation is not commutative:

$$h \circ g \neq g \circ h$$

For bijective functions f, we have

$$f^{-1} \circ f = \mathrm{id}_X,\ f \circ f^{-1} = \mathrm{id}_Y$$

where id_X is the identity function $\mathrm{id}_X : X \to X$.

For surjective functions f we may cancel from the right:

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2$$

For injective functions f we may cancel from the left:

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

Q.19: $f(f(x)) = x + 1$. Show that $f(x+1) = f(x) + 1$

Functional equations

Functional equations are very common in competitive mathematics as they may require a broad set of skills. Solutions are often very similar to how mathematicians perform real research. I therefore encourage you to approach functional equations as exactly that; a scientist. Try to explore the functions, building up a knowledge base, until you feel like you understand the function. Then you can start trying to prove a solution. Once you have finished your proof and found a solution you must **verify that your found solution satisfies the given equation**. This last step is absolutely crucial and often costs students many valuable points and sometimes medals in IMO. In this section, we will approach the following problem together:

Problem 2: Find all real functions $f(x)$ such that

$$f(f(x-y)) = f(x)f(y) - f(x) + f(y) - xy$$

for all $x, y \in \mathbb{R}$

A good start when approaching functional equations of this type is to try to find points where the equation simplify.

Q.20: Set $c = f(0)$, find $f(c)$.

Q.21: Assume f is surjective, find all real functions $f(x)$.

Q.22: Show by proof of contradiction that $c = 0$.

Hint: first show that $c \neq 0 \implies t((c-2)t + 2(c-1)) = 0$, where t is in the image of f (i.e. a value taken by $f(x)$).

Q.23: We have now shown that $f(t) = -t$ when t is in the image of f. Use this to recover $f(x), \forall\ x \in \mathbb{R}$

This problem trains us to be careful as there are many instances where a lazy student may declare that they have found a solution when in fact they have only found a part. It is easy to accidentally conclude that $f(x) = kx + m$, or that $f(x)^2 = x^2 \implies f(x) = \pm x$. By stepping through the problem slowly and acquiring increasing knowledge, we were able to solve the problem for a full score. Or were we..!?

Q.24: Verify that $f(x) = -x$ solves Problem 2.

that's it.

Let's finish with a very famous functional equation, the solution to which you *may* get away with referencing on an exam:

Problem 3: Cauchy's equation
This is a famously complicated equation. Here, we will tackle the much simpler case where we assume f is continuous. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Find f such that

$$f(x + y) = f(x) + f(y)$$

Problem 4:
Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + y) = f(x) + f(y) + xy$$

for all $x, y \in \mathbb{Q}$.

Some tricks

- Try to find a solution to start with. Test $f_0(x) = 0$, $f_0(x) = \pm x$, $f_0(x) = cx$, $f_0(x) = c$, $f_0(x) = x+c$. One of these are very often the solution in competitive mathematics, and it is nice to have some direction before you start. However, if you find a solution, don't assume f will be nice.
- If you find some candidate $f_0(x)$ that solves the equation, you may attempt to set $g(x) = f(x) - f_0(x)$ or $f(x) = h(x)f_0(x)$ and attempt to show that $g(x) = 0$ or $h(x) = 1$.
- Study $f(0)$ or some $f(a) = 0$ if these exist. Use substitutions.
- If the equation contains symmetries, $x = y$ may get you far.
- If you know $f(f(x))$, then applying the function once more could be useful (see Q.19).
- Try to make things cancel.
- try to understand what type of function you are working with. Can we show that it is injective or surjective?
- Carefully document all your findings on a separate paper, clearly stating any assumptions you made when arriving at a result.
- Solve many problems. You really only get good at solving functional equations by exposing yourself to many problems. More so for functional equations than most domains in competitive mathematics.

Problems

Problem 5:
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such thats

$$f(xy + x) = x\,f(y) + f(x)$$

for all $x, y \in \mathbb{R}$.

Problem 6: [David Yang]
Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y) = f(x^{27} + 2y) + f(x^4)$$

for all $x, y \in \mathbb{R}$.

Problem 7: [Kyrgyzstan 2012]
Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f\big(f(x)^2 + f(y)\big) = x\,f(x) + y$$

for all $x, y \in \mathbb{R}$.

Problem 8:
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 1$ and

$$f(xy + f(x)) = xf(y) + f(x)$$

Problem 9:
Find all $f : \mathbb{R}\setminus \{-1, 1\} \to \mathbb{R}$ such that

$$f\left(\frac{x-3}{1+x}\right) + f\left(\frac{x+3}{1-x}\right) = x$$

Problem 10: [Canada 2002]
Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Determine all functions $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$x\,f(y) + y\,f(x) = (x + y)\,f(x^2 + y^2)$$

for all $x, y \in \mathbb{N}_0$.

Problem 11:
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Find f such that

$$f(x + y) + f(x - y) = 2f(x) + 6xy^2$$

for all $x, y \in \mathbb{R}$

Problem 12:
A *reflection* is a bijective function $f : \mathbb{M} \to \mathbb{M}$ for which $f(f(x)) = x,\ \forall x \in \mathbb{M}$. Prove that for any bijective function $F : \mathbb{M} \to \mathbb{M}$, there exists two reflections f_1, f_2 so that $F = f_1 \circ f_2\ \forall x \in \mathbb{M}$.

Problem 13:
The set F consists of all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(3x) \geq f(f(2x)) + x,\ \forall x \in \mathbb{R}^+$$

Find the largest real constant a such that

$$f(x) \geq ax,\ \forall f \in F,\ \forall x \in \mathbb{R}^+$$

Problem 14:
Find all bounded continuous $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)^2 - f(y)^2 = f(x + y)f(x - y)$$

for all $x, y \in \mathbb{R}$

Hints (Problems)

Problem 1. Avoid the temptation to answer $f(x) = 1$. Try factoring the equation and remember monotone may mean decreasing *or* increasing.

Problem 2.

Problem 3. The rational numbers are dense on $\mathbb{R}$. If we can find $f(r)$ for some $r \in \mathbb{Q}$, then by continuity we know $f(r) = f(x)$

Problem 4. Set $x = y = 0$, then define $g(x) = f(x) - \frac{x^2}{2}$.

Problem 5. Set $y = 0$, then set $y = -1$.

Problem 6. Choose a y so the annoying terms cancel

Problem 7. Set all variables to zero to find a value u with $f(u) = 0$, then substitute $x = u$ to discover that f is an involution.

Problem 8. Try $x = 1$, then $y = 0$. Induction.

Problem 9. set $t = (x-3)/(x+1)$. Then set $t = (x+3)/(1-x)$. Then add.

Problem 10. Try setting $x = y$ and $y = 0$ first. Define $g(x) = f(x) - f(0)$ and show g satisfies a relation forcing it to be identically zero.

Problem 11. Set $u = x + y,\ v = x - y$. WLOG $f(0) = 0$

Problem 12. Consider applying F any times to the same x

Problem 13.

Problem 14. This is hard. Start by trying to find one solution

Solutions (Problems)

Problem 1. Writing $(f(x) - 1)(f(x) + 1) = 0$, we find $f(x) = \pm 1$. However, we do not know *when* f takes which value. There are still an infinite number of solutions. Say WLOG f is increasing (for decreasing just use $-f$). If f is not constant, for some $a, f(a) = 1$ and $f(x) = 1, x > a$, and for some $b, f(b) = -1$, and $f(x) = -1, x < b$. The breakpoint is the supremum $c = \sup\{x | f(x) = -1\}$ which must exist if f is non constant. However, we do not know if the supremum is in the set $x : f(x) = -1$ or outside it. Both possibilities are possible. We end up with the cases

$$f_1(x) = 1, \qquad f_{-1}(x) = -1$$

$$f_c^+(x) = \begin{cases} 1, & \text{if } x \geq c \\ -1, & \text{if } x < c \end{cases} \quad f_c^-(x) = \begin{cases} 1, & \text{if } x > c \\ -1, & \text{if } x \leq c \end{cases}$$

As well as $-f_c^+(x), -f_c^-(x)$.

Problem 2. Solution in lecture

Problem 3. Set $x = y = 0$. Induction gives $f(nx) = nf(x)\ \forall x \in \mathbb{R},\ n \in \mathbb{N}$. We can show $f(-nx) = nf(-x) = -nf(x)$. It also follows that $f(x) = f(m\frac{x}{m}) = mf(x/m)$. Hence, $f(r) = rf(x)$, $\forall x \in \mathbb{R},\ r \in \mathbb{Q}$. Setting $f(1) = a$, we get $f(r) = ar$ which by the hint gives us $f(x) = ax$

Problem 4. Setting $x = y = 0$: $f(0) = 0$. Define $g(x) = f(x) - \frac{x^2}{2}$. Substituting:

$$g(x+y) + \tfrac{(x+y)^2}{2} = g(x) + \tfrac{x^2}{2} + g(y) + \tfrac{y^2}{2} + xy,$$

which simplifies to $g(x+y) = g(x) + g(y)$. This is Cauchy's equation over $\mathbb{Q}$, so $g(x) = cx$ for some $c \in \mathbb{Q}$.

Problem 5. Set $y = 0$: $f(x) = x\,f(0) + f(x)$, so $f(0) = 0$. Set $y = -1$: $f(0) = x\,f(-1) + f(x)$, hence $f(x) = -f(-1) \cdot x$. So $f(x) = cx$ where $c = -f(-1) = f(1)$

Problem 6. $f \equiv 0$. Setting $x = y = 0$ gives $f(0) = 2f(0)$, so $f(0) = 0$. Now set $y = x^2 - x^{27}$ so that $x^2 + y = x^{27} + 2y$. Then both f-arguments on the left and first term on the right coincide, giving

$$f(x^2 + (x^2 - x^{27})) = f(x^2 + (x^2 - x^{27})) + f(x^4),$$

hence $f(x^4) = 0$ for all $x \in \mathbb{R}$. Since x^4 ranges over all of $[0, \infty)$, we have $f \equiv 0$ on $[0, \infty)$.

Finally, setting $y = 0$: $f(x^2) = f(x^{27}) + f(x^4)$. The left side and $f(x^4)$ are both zero for all x, so $f(x^{27}) = 0$ for all x. Since $x \mapsto x^{27}$ is a bijection on $\mathbb{R}$, we get $f \equiv 0$ on all of $\mathbb{R}$.

Problem 7. The answers are $f(x) = x$ and $f(x) = -x$. Set $x = y = 0 \implies f(f(0)^2 + f(0)) = 0$, so there exists u with $f(u) = 0$. Setting $x = u$: $f(f(y)) = y$ for all y, i.e. f is an involution, hence bijective.

Substituting $x = f(t)$ (so $f(x) = f(f(t)) = t$) into the original equation:

$$f(t^2 + f(y)) = f(t) \cdot t + y.$$

But the original with $x = t$ gives $f(f(t)^2 + f(y)) = t\, f(t) + y$. Since both right-hand sides equal $t\, f(t) + y$, and f is injective:

$$t^2 + f(y) = f(t)^2 + f(y) \implies t^2 = f(t)^2 \implies f(t) = \pm t$$

for each t.

It remains to exclude "mixed" solutions. Suppose $f(a) = a$ and $f(b) = -b$ with $a, b \neq 0$. Substituting $x = a, y = b$ into the original: $f(a^2 - b) = a^2 + b$, but also $a^2 - b = \pm(a^2 + b)$, forcing $b = 0$, a contradiction. So $f(x) = x$ for all x, or $f(x) = -x$ for all x. Both are easily verified.

Problem 8. Testing, we find $f(1) = f(0) + 1$ and by induction $f(n) = n,\ n \in \mathbb{Z}$. Let's try to show that $f(x) = x$ is a general solution. Testing, we find $f(f(x)) = f(x)$. $f(x) = y \implies f((x+1)f(x)) = xf(x) + f(x)$, $f(y) = x \implies f((y+1)f(y) = f(y)^2 + f(y)$. Note we can say this since f has the full real axis as both its' domain and codomain. Combine to get $f(x)(x - f(x)) = 0 \implies f(x) = x$ or $f(x) = 0$. Set $f(a) = 0$ and you quickly find $a = 0 = f(a)$. Therefore $f(x) = x$ is the only solution.

Problem 9.

Problem 10. The answer is $f \equiv c$ for any constant $c \in \mathbb{N}_0$.

Setting $y = 0$ gives $x\, f(0) = x\, f(x^2)$, so $f(x^2) = f(0)$ for all $x \geq 1$. Setting $x = y$ gives $f(x) = f(2x^2)$. Since $f(2x^2) = f((2x^2)^2) = \ldots$ is eventually f of a perfect square, and f of any perfect square equals $f(0)$, we might hope f is constant. To make this precise, set $g(x) = f(x) - f(0)$, so $g(x^2) = 0$ for all $x \geq 1$.

The original equation can be rewritten as

$$(x + y)\, g(x^2 + y^2) = x\, g(y) + y\, g(x). \qquad (*)$$

We want to show $g \equiv 0$. Suppose for contradiction some $g(x) \neq 0$. For any even $x > 4$, set $y = \frac{x^2}{4} - 1$; then $x^2 + y^2$ is a perfect square, so the left side of $(*)$ is zero, giving $x\, g(y) = -y\, g(x)$. Similarly, for odd $x > 3$, one can find $y > x$ with $x^2 + y^2$ a perfect square.

Starting from any $x_0 > 4$ with $g(x_0) \neq 0$, we construct a sequence $x_0, x_1, x_2, \ldots$ where each $x_i^2 + x_{i+1}^2$ is a perfect square, $x_{i+1} > x_i$, and $g(x_{i+1}) = -\frac{x_i}{x_{i+1}}\, g(x_i)$. The signs of $g(x_i)$ alternate, and since g is integer-valued, $|g(x_i)| \geq 1$ for all i. But then $g(x_i) < -f(0)$ for large enough i, meaning $f(x_i) < 0$, contradicting $f\colon \mathbb{N}_0 \to \mathbb{N}_0$.

Problem 11.

Problem 12. Consider the set reached by applying F many times to x: $\mathbb{M}_x = \{F^k(x) \mid k \in \mathbb{Z}\}$ where $F^2 = F \circ F$. We see that for any two $x, y \in \mathbb{M}$, either $\mathbb{M}_x = \mathbb{M}_y$ or $\mathbb{M}_x \cap \mathbb{M}_y = \emptyset$. We conclude that $\mathbb{M}$ is the union of pairwise disjoint sets and that it is sufficient to solve for only $\mathbb{M} = \mathbb{M}_x$. Clever construction now gives us the example $f_1 : F^k(x) \to F^{1-k}(x),\ f_2 : F^k(x) \to F^{-k}(x)$ which satisfies

$$f_1(f_2(F^k(x))) = f_1(F^{-k}(x)) = F^{1+k}(x) = F(F^k(x))$$

which is an example of existance.

Problem 13. a=1/2

Problem 14. $f(x) = a \sin bx,\ \forall a, b \in \mathbb{R}$

Solutions (Questions)

Q.1. A function for which the equation $f(x) = y$ has *at least* one solution.

Q.2. A function for which the equation $f(x) = y$ has *at most* one solution.

Q.3. A function for which the equation $f(x) = y$ has *exactly* one solution.

Q.4. a) injective, b) bijective, c) general, d) surjective

Q.5.

Q.6. f_1: neither, f_2: surjective, f_3: injective, f_4: bijective

Q.7. f_1: neither, f_2: surjective, f_3: injective, f_4: bijective

Q.8. 0

Q.9. $\inf f = 0$, $\sup f = 1$, no minimum or maximum

Q.10. E.g. $f(x) = \frac{x^2}{1+x^2}$

Q.11. $\inf f = \min f = 0$, attained at $x = 0$; restricting to $(0, \infty)$ preserves $\inf f = 0$ but destroys the minimum

Q.12. True. If $\min f = m$ then m is attained and is a lower bound, so $m = \inf f$.

Q.13. E.g., $f(x) = 1/x$ on $(0, 1)$

Q.14. $x = 2$

Q.15. $x = \{-1, 1\}$

Q.16. No maximum

Q.17.

Q.18. 0. We need to also check all non-differentiable points.

Q.19. Substitute $u = f(x)$

Q.20. Set $x = y$, $f(c) = c^2$

Q.21. Set $y = 0$, $f(x) = t$, $f(t) = (c - 1)t + c$

Q.22. Combine the solutions above: $f(c) = c^2 = f(x)^2 - x^2, f(t) = (c-1)t+c$, setting $x = t$. Then, $t = 0$ or $t = -2(c-1)/(c-2)$ which implies c is the only non-zero value attained by f. $f(c) = c^2 = c \implies c = 1$. But then $t = f(x) = 0 = c$ which is a contradiction.

Q.23. We have $f(x)^2 = x^2$ which may give us $f(x) = \pm x$. Assume there is some $f(x^*) = x^* \neq 0$. Then, per definition, x^* is in the image of f, and for all such x^* we have already shown $f(t) = -t$. $f(x) = -x$ is therefore the only solution.

Q.24.